

PROBLEM SECTION

Edited by Nenad P. Cakić

We publish research problems in all areas of Mathematics that fall in one of the following categories:

- a) Research problems or conjectures with a solution not known to the proposer. These problems are marked with an asterisk after their number.
- b) Research problems seeking a new, more elegant solution
- c) Inquiries about references and state of the art regarding a particular problem.

Problems should be submitted in a form that is easy to understand to a nonspecialist in a field. If using special terms or notations can not be avoided, they should be defined in a statement of a problem. A problem may be accompanied by a short comment (addressed primarily to specialists) that explains why the solution could be of an interest.

Solutions should be worked out in all reasonable details.

Correspondence regarding Problem Section should be sent to:

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- **Problem 78*** proposed by MICHAEL TH. RASSIAS, National Technical University of Athens, Greece.

Conjecture. *For every prime number p greater than two there are two primes p_1, p_2 ($p_1 < p_2$) such that*

$$p = \frac{p_1 + p_2 + 1}{p_1}.$$

The Conjecture can be also stated in the following way: *For every prime number p greater than two there are two primes p_1, p_2 ($p_1 < p_2$) such that the numbers $(p-1)p_1, p_2$ are consecutive integers.*

NOTE. MICHAEL TH. RASSIAS was awarded with a Silver Medal from the International Mathematical Olympiad, Tokyo, Japan in 2003.

- **Problem 79** proposed by OVIDIU FURDUI, Kalamazoo, USA.

Let n, k be natural numbers, $n, k \geq 1$ and m be a real number such that $m > n$ and $n > \frac{m}{2k+1}$. Prove that

$$\begin{aligned} & \int_0^{+\infty} \int_0^{+\infty} \cdots \int_0^{+\infty} \frac{(\sin x_1 \sin x_2 \cdots \sin x_n)^{2k}}{(x_1 + x_2 + \cdots + x_n)^{2n}} dx_1 dx_2 \cdots dx_n \\ &= \frac{((2k)!)^n}{\Gamma(m)} \int_0^{+\infty} \frac{t^{m-n-1}}{\left((t^2 + (2k)^2)(t^2 + (2(k-1))^2) \cdots (t^2 + (2 \cdot 1)^2) \right)^n} dt, \end{aligned}$$

where Γ is the Gamma function.

- **Problem 80*** proposed by MEHDI HASSANI, Zenjan, Iran.

Consider an even number of points on the plane. Show that there is a line in which divide these points (a half in one side and the others in the other side of line) and sum of distance of points in every side from line are equal.

Note that when all points are on a line, sum of distances in both sides are equal to zero.

- **Problem 81** proposed by Y. N. ALIYEV, Baku, Azerbaijan.

Do there exist irrational numbers $a > 1, b > 1$ for which the inequality

$$|a^n - b^m| > 2006$$

is true for all positive integers n, m ?

- **Problem 82*** proposed by WALTHER JANOUS, Innsbruck, Austria.

Let a, b and c be the sides of an arbitrary triangle and $p > 1$ be an arbitrary real number. Prove that the following inequality is valid

$$(a^p + b^p)^{1/p} + (b^p + c^p)^{1/p} + (c^p + a^p)^{1/p} < \frac{2 + 2^{1/p}}{2} (a + b + c).$$

Show furthermore that the factor on the right-hand side is best possible.

- **Problem 83*** proposed by SLAVKO SIMIĆ, Belgrade, Serbia.

Let f be a non-constant entire function with non-negative TAYLOR coefficients. Prove that

$$\lim_{r \rightarrow +\infty} \frac{f(r)}{r f'(r)}$$

always exists and represents a rational number.

- **Problem 84** proposed by PÉTER IVÁDY, Budapest, Hungary.

Show, that for $0 < a < b < +\infty$,

$$\frac{a+b}{2} - \sqrt{ab} < \frac{1}{8} \frac{(a-b)^2(a+4b)(b+4a)}{(a+b)(a^2+8ab+b^2)} < \sqrt{\frac{a^2+b^2}{2}} - \frac{a-b}{\ln a - \ln b}$$

holds.

- **Problem 85*** proposed by MIRKO JOVANOVIĆ, Belgrade, Serbia.

Let $(a_n)_{n \in \mathbb{N}}$ be a real bounded sequence, then

$$\lim_{n \rightarrow +\infty} \left(a_{n+k} - \sum_{i=0}^{k-1} \lambda_i a_{n+i} \right) = \alpha \Leftrightarrow \lim_{n \rightarrow +\infty} a_n = \alpha \left(1 - \sum_{i=0}^{k-1} \lambda_i \right)^{-1},$$

where k is fixed positive integer and $\sum_{i=0}^{k-1} \lambda_i < 1$, $\lambda_i \geq 0$ ($i = 0, 1, \dots, k-1$).

- **Problem 86*** proposed by SLAVKO SIMIĆ, Belgrade, Serbia.

Let

$$f_1 := x + y + z; \quad f_2 := xy + xz + yz; \quad f_3 := xyz,$$

be elementary symmetric functions with real variables x, y, z .

Prove the following inequality:

$$16 f_1^3 f_3 + 16 f_2^3 + 9 f_3^2 \leq 34 f_1 f_2 f_3 + 7 f_1^2 f_2^2.$$

SOLUTIONS

- **Problem 60** proposed by MIHÁLY BENCZE and MARIAN DINCA, Brasov, Romania.

If $f : A \rightarrow (0, +\infty)$ ($A \subseteq \mathbb{R}$) is a concave function and

$$0 < \{ \max f(x) \mid x \in A \} \leq M/2,$$

then

$$\left(\frac{\prod_{k=1}^n (f(x_k))^{\lambda_k}}{\prod_{k=1}^n (M - f(x_k))^{\lambda_k}} \right)^{1/\sum_{k=1}^n \lambda_k} \leq \frac{f \left(\frac{\sum_{k=1}^n \lambda_k x_k}{\sum_{k=1}^n \lambda_k} \right)}{M - f \left(\frac{\sum_{k=1}^n \lambda_k x_k}{\sum_{k=1}^n \lambda_k} \right)},$$

where $\lambda_k > 0$, $x_k \in A$ ($k = 1, 2, \dots, n$).

Letter to the editor:

Flaw in the solution of Problem 60.

Dear Editor Cakić,

I just browsed the Problem Section of Vol. 17/2006 issue, and noticed a flaw in the published solution of Problem 60 (126–128). The solution assumes that f is differentiable (which is not assumed in the original problem) and applied chain rule to g . The solution I submitted did not need this assumption and proved the convexity of g in a more subtle way. Thanks for your attention.

With kind regards, Li

Solution by LI ZHOU, Winter Haven, USA.

Define $g(u) = \log \frac{u}{M-u}$ for $0 < u \leq M/2$ and $h(x) = g(f(x))$ for $x \in A$. Then $g'(u) = \frac{M}{u(M-u)} > 0$ and $g''(u) = \frac{-M(M-2u)}{u^2(M-u)^2} \leq 0$. Now suppose that $x, y, z \in A$ and $x < y < z$. Then

1) if $f(x) < f(z) < f(y)$, then $\frac{f(z) - f(x)}{z - x} \geq \frac{f(y) - f(z)}{y - z} > 0$ and

$$\frac{g(f(z)) - g(f(x))}{f(z) - f(x)} \geq \frac{g(f(y)) - g(f(z))}{f(y) - f(z)} > 0;$$

2) if $f(x) > f(z) > f(y)$, then $0 > \frac{f(z) - f(x)}{z - x} \geq \frac{f(y) - f(z)}{y - z}$ and

$$0 < \frac{g(f(z)) - g(f(x))}{f(z) - f(x)} \leq \frac{g(f(y)) - g(f(z))}{f(y) - f(z)};$$

3) if $f(x) < f(z)$ and $f(y) < f(z)$, then $\frac{f(z) - f(x)}{z - x} > 0 > \frac{f(y) - f(z)}{y - z}$, $\frac{g(f(z)) - g(f(x))}{f(z) - f(x)} > 0$ and $\frac{g(f(y)) - g(f(z))}{f(y) - f(z)} > 0$.

Hence, in all these three cases,

$$\begin{aligned} \frac{h(z) - h(x)}{z - x} &= \frac{f(z) - f(x)}{z - x} \frac{g(f(z)) - g(f(x))}{f(z) - f(x)} \\ &\geq \frac{f(y) - f(z)}{y - z} \frac{g(f(y)) - g(f(z))}{f(y) - f(z)} = \frac{h(y) - h(z)}{y - z}. \end{aligned}$$

Also, it is easier to see that $\frac{h(z) - h(x)}{z - x} \geq \frac{h(y) - h(z)}{y - z}$ in the more trivial cases of $f(x) = f(z)$ or $f(z) = f(y)$. Hence h is concave. By JENSEN's inequality,

$$\sum_{k=1}^n \frac{\lambda_k}{\lambda_1 + \dots + \lambda_n} h(x_k) \leq h\left(\sum_{k=1}^n \frac{\lambda_k x_k}{\lambda_1 + \dots + \lambda_n}\right),$$

which is equivalent to the desired inequality.

COMMENT. In fact, we have established the following useful result: 1) if f is concave and g is increasing and concave, then $g \circ f$ is concave. Similarly, 2) if f is concave and g is decreasing and convex, then $g \circ f$ is convex; 3) if f is convex and g is increasing and convex, then $g \circ f$ is convex; 4) if f is convex and g is decreasing and concave, then $g \circ f$ is concave.

Problems 62 and 63*, proposed by JOSE LUIS DÍAZ-BARRERO, Barcelona, Spain and MIHÁLY BENCZE, Brasov, Romania, respectively.

Let n be a positive integer. Prove that

$$\prod_{k=1}^n \binom{n}{k}^{k^3} \leq \left(\frac{2^{n-1}(n+3)}{(n+1)^2} \right)^{\binom{n+1}{2}^2}.$$

When does the equality occur?

If $S_n^p = \sum_{k=1}^n k^p$ and $R_n^p = \sum_{k=1}^n k^p \binom{n}{k}$, where $n, p \in \mathbb{N}$, then

$$\prod_{k=1}^n \binom{n}{k}^{k^p} \leq \left(\frac{R_n^p}{S_n^p} \right)^{S_n^p}.$$

This is a generalization of the Problem 62.

Solution by INGRAM OLKIN, Stanford, USA.

Problem 63 is essentially the arithmetic-geometric mean inequality. Rewrite the inequality as follows:

Define $\binom{n}{k} = a_k$ and $\frac{k^p}{S_n^p} = p_k$. Note that $0 \leq p_k \leq 1$, $\sum_{k=1}^n p_k = 1$. Then the inequality in Problem 63 can be written as

$$\prod_{k=1}^n a_k^{p_k} \leq \sum_{k=1}^n p_k a_k,$$

which is the arithmetic-geometric mean inequality.

Also solved by LI ZHOU (Problem 62), WALTER JANOUS (Problems 62 and 63* - solution similar to the OLKIN solution), MICHEL BATAILLE (Problem 62) and the proposer (Problem 62).

• **Problem 64** proposed by MIHÁLY BENCZE, Brasov, Romania.

If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a convex function, then for all $x_k \in \mathbb{R}$ ($k = 1, 2, \dots, n$) holds the inequality

$$2 \cdot \sum_{k=1}^n f(x_k^2) \geq f(x_1 x_2) + f(x_2 x_3) + \dots + f(x_n x_1) \\ + f(x_1^2 - x_1 x_2 + x_2^2) + f(x_2^2 - x_2 x_3 + x_3^2) + \dots + f(x_n^2 - x_n x_1 + x_1^2).$$

Solution by LI ZHOU, Winter Haven, USA.

For the inequality to hold, it is necessary to require that $x_1, \dots, x_n \geq 0$. For a counter example, take $f(x) = x^2$, $n = 2$, $x_1 = 1$, $x_2 = -1$. Now for any pair (x, y) with $x \geq y \geq 0$, $x^2 \geq x^2 - xy + y^2 \geq xy \geq y^2$, and thus $(x^2, y^2) \succ (x^2 - xy + y^2, xy)$. Hence, by the majorization inequality,

$$f(x^2) + f(y^2) \geq f(x^2 - xy + y^2) + f(xy).$$

The proof is complete upon summing all such inequalities over pairs $\{x_1, x_2\}$, $\{x_2, x_3\}, \dots$, and $\{x_n, x_1\}$.

Also solved by M. ZEKI SARIKAYA and BASAK KARPUZ, Afyon, Turkey (with same reverse example), SLAVKO SIMIĆ, Belgrade, Serbia (counter example: $f(x) = x^2$, $x_1 = x_2 = \dots = x_{n-1} = 1$, $x_n = x < 0$) and partially by the proposer.

• **Problem 65** proposed by MIHÁLY BENCZE, Brasov, Romania.

(i) If $a, b, c \geq 0$ then

$$\begin{aligned} (a^2 + b)(a + b^2)(b^2 + c)(b + c^2)(c^2 + a)(c + a^2) \\ \geq abc(a + 1)(b + 1)(c + 1)(a + bc)(b + ac)(c + ab). \end{aligned}$$

(ii)* If $a_k \geq 0$ ($k = 1, 2, \dots, n$) and $n \geq 4$, then

$$\begin{aligned} (a_1^2 + a_2)(a_1 + a_2^2)(a_2^2 + a_3)(a_2 + a_3^2) \cdots (a_n^2 + a_1)(a_n + a_1^2) \\ \geq (a_1 + a_2 a_3)(a_2 + a_3 a_4) \cdots (a_n + a_1 a_2) \cdot \prod_{k=1}^n a_k (1 + a_k). \end{aligned}$$

Solution (part (i)) by SLAVKO SIMIĆ, Belgrade, Serbia.

Since $(x^2 + y)(y + z^2) = (y + xz)^2 + y(x - z)^2$, and

$$(x + y^2)(y + x^2) = xy(x + 1)(y + 1) + (x + y)(x - y)^2,$$

then for non-negative x, y, z , we get

$$(1) \quad (x^2 + y)(y + z^2) \geq (y + xz)^2,$$

and

$$(2) \quad (x + y^2)(y + x^2) \geq xy(x + 1)(y + 1).$$

Now, from (1) it follows that

$$\begin{aligned} (a^2 + b)(b + c^2) &\geq (b + ac)^2, \\ (b^2 + c)(c + a^2) &\geq (c + ab)^2, \\ (c^2 + a)(a + b^2) &\geq (a + bc)^2, \end{aligned}$$

and from (2), we get

$$\begin{aligned}(a + b^2)(b + a^2) &\geq ab(a + 1)(b + 1), \\ (b + c^2)(c + b^2) &\geq bc(b + 1)(c + 1), \\ (c + a^2)(a + c^2) &\geq ac(a + 1)(c + 1).\end{aligned}$$

Multiplying those inequalities we obtain the desired result.

LI ZHOU found a different generalization of (i):

If $a_k \geq 0$ ($k = 1, 2, \dots, n$) and $n \geq 2$, then

$$\prod_{k=1}^n (a_k^2 + a_{k+1})(a_k + a_{k+1}^2) \geq \prod_{k=1}^n (a_k + a_{k-1}a_{k+1})a_k(1 + a_k),$$

where the indices are taken modulo n .

Firstly, for $1 \leq k \leq n$,

$$(a_k^2 + a_{k+1})(a_k + a_{k+1}^2) - a_k a_{k+1}(1 + a_k)(1 + a_{k+1}) = (a_k + a_{k+1})(a_k - a_{k+1})^2 \geq 0.$$

Hence

$$\sqrt{(a_k^2 + a_{k+1})(a_k + a_{k+1}^2)} \geq \sqrt{a_k a_{k+1}(1 + a_k)(1 + a_{k+1})}.$$

Secondly,

$$(a_{k-1}^2 + a_k)(a_k + a_{k+1}^2) - (a_k + a_{k-1}a_{k+1})^2 = a_k(a_{k-1} - a_{k+1})^2 \geq 0.$$

Hence

$$\sqrt{(a_{k-1}^2 + a_k)(a_k + a_{k+1}^2)} \geq a_k + a_{k-1}a_{k+1}.$$

Multiplying these $2n$ equalities completes the proof.

Also solved by WALTER JANOUS, Innsbruck, Austria and by proposer.

- **Problem 66** proposed by MIHÁLY BENCZE, Brasov, Romania.

If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a convex and integrable function, then holds the following inequality:

$$\int_0^{\operatorname{sh} 1} f(x) dx + \int_1^{\operatorname{ch} 1} f(x) dx \geq \int_0^1 e^x f(e^{-x} \operatorname{sh} 2x) dx.$$

Solution by MICHEL BATAILLE, Rouen, France.

Let $L = \int_0^{\operatorname{sh} 1} f(x) dx + \int_1^{\operatorname{ch} 1} f(x) dx$. The substitution $x = \operatorname{sh} t$ in the first integral and $x = \operatorname{ch} t$ in the second one give

$$L = \int_0^1 (\operatorname{ch} t f(\operatorname{sh} t) + \operatorname{sh} t f(\operatorname{ch} t)) dt.$$

Now, the convexity of the f yields

$$\begin{aligned} \operatorname{ch} t \cdot f(\operatorname{sh} t) + \operatorname{sh} t \cdot f(\operatorname{ch} t) &= e^t \left(\frac{\operatorname{ch} t}{e^t} f(\operatorname{sh} t) + \frac{\operatorname{sh} t}{e^t} f(\operatorname{ch} t) \right) \\ &\geq e^t \left(f \left(\frac{\operatorname{ch} t \operatorname{sh} t}{e^t} + \frac{\operatorname{sh} t \operatorname{ch} t}{e^t} \right) \right) \\ &= e^t f \left(\frac{\operatorname{sh} 2t}{e^t} \right) = e^t f(e^{-t} \operatorname{sh} 2t). \end{aligned}$$

Integrating from 0 to 1, we obtain $L \geq \int_0^1 e^t f(e^{-t} \operatorname{sh} 2t) dt$.

Also solved by OVIDIN FURDUI, Kalamazoo, MI, USA, LI ZHOU, Winter Haven, USA, M. ZEKI SARIKAYA and BASAK KAPRUZ, Afyon, Turkey, SLAVKO SIMIĆ, Belgrade, Serbia and the proposer.

• **Problem 67** proposed by JOSÉ LUIS DÍAZ-BARRERO, Barcelona, Spain.

Suppose that the roots of the equation $A(z) = z^n + \sum_{k=0}^{n-1} a_k z^k = 0$ ($a_0 \neq 0$) are the distinct nonzero complex numbers z_1, z_2, \dots, z_n . Prove that

$$\sum_{k=1}^n \frac{1}{z_k} \left(\prod_{j=1, j \neq k}^n \frac{1}{z_k - z_j} + \prod_{j=1, j \neq k}^n \frac{z_j}{z_k - z_j} \right) = \frac{(-1)^n a_1 - 1}{a_0}.$$

Solution by MICHEL BATAILLE, Rouen, France.

Let the polynomial $A_k(z)$ be defined by $A(z) = (z - z_k)A_k(z)$, ($k = 1, 2, \dots, n$). Differentiation gives $A'(z) = A_k(z) + (z - z_k)A'_k(z)$ and so $A'(z_k) = A_k(z_k)$. Note that the latter is not 0 since the roots of $A(z)$ are simple. Now, consider the decomposition of $\frac{1}{A(z)}$ into partial fractions:

$$\frac{1}{A(z)} = \frac{\alpha_1}{z - z_1} + \frac{\alpha_2}{z - z_2} + \dots + \frac{\alpha_n}{z - z_n}.$$

For $k = 1, 2, \dots, n$, we have $\alpha_k = \left[\frac{z - z_k}{A(z)} \right]_{z=z_k} = \frac{1}{A_k(z_k)} = \frac{1}{A'(z_k)}$ and so

$$(1) \quad \frac{1}{A(z)} = \frac{1}{A'(z_1)} \cdot \frac{1}{z - z_1} + \dots + \frac{1}{A'(z_n)} \cdot \frac{1}{z - z_n}.$$

It follows that

$$(2) \quad \frac{-A'(z)}{(A(z))^2} = \frac{1}{A'(z_1)} \cdot \frac{-1}{(z - z_1)^2} + \dots + \frac{1}{A'(z_n)} \cdot \frac{-1}{(z - z_n)^2}.$$

On the other hand, since $A'(z) = \sum_{k=1}^n \prod_{j=1, j \neq k}^n (z - z_j)$, we have $A'(z_k) = \prod_{j=1, j \neq k}^n (z_k - z_j)$. Therefore

$$T_1 := \sum_{k=1}^n \frac{1}{z_k} \prod_{j=1, j \neq k}^n \frac{1}{z_k - z_j} = \sum_{k=1}^n \frac{1}{z_k} \cdot \frac{1}{A'(z_k)} = \frac{-1}{A(0)} = \frac{-1}{a_0} \quad (\text{using (1)}),$$

and

$$\begin{aligned} T_2 &:= \sum_{k=1}^n \frac{1}{z_k} \prod_{j=1, j \neq k}^n \frac{z_j}{z_k - z_j} = (z_1 z_2 \cdots z_n) \sum_{k=1}^n \frac{1}{z_k^2} \prod_{j=1, j \neq k}^n \frac{1}{z_k - z_j} \\ &= (-1)^n a_0 \sum_{k=1}^n \frac{1}{A'(z_k)} \cdot \frac{1}{z_k^2} = (-1)^n a_0 \frac{A'(0)}{(A(0))^2} = (-1)^n \frac{a_1}{a_0} \quad (\text{using (2)}). \end{aligned}$$

The desired result $T_1 + T_2 = \frac{(-1)^n a_1 - 1}{a_0}$ follows immediately.

Also solved by LI ZHOU, Winter Haven, USA and the proposer.

• **Problem 68** proposed by BRANKO MALEŠEVIĆ, Belgrade, Serbia.

For alternating sum of factorial [1] (problem B 43): $A_n = \sum_{i=1}^n (-1)^{n-i} i!$ prove that the following congruence is true:

$$A_p \equiv \left([e(p! - (p-1)!)] + 1 \right) \pmod{p},$$

where p is a prim number and e base of the natural logarithm.

Reference

1. R. K. GUY: *Unsolved problems in number theory*. 2nd. ed., 1994, Springer Verlag.

Solution by LI ZHOU, Winter Haven, USA.

Firstly,

$$A_p = \sum_{i=1}^p (-1)^{p-i} i! \equiv (-1)^p \sum_{i=1}^{p-1} \frac{(p-1)!}{(p-1-i)!} \equiv - \sum_{k=0}^{p-2} \frac{(p-1)!}{k!} \pmod{p}.$$

On the other hand,

$$\begin{aligned} e(p! - (p-1)!) &- \left(p! \sum_{k=0}^p \frac{1}{k!} - (p-1)! \sum_{k=0}^{p-1} \frac{1}{k!} \right) = p! \sum_{k=p+1}^{\infty} \frac{1}{k!} - (p-1)! \sum_{k=p}^{\infty} \frac{1}{k!} \\ &= \sum_{k=p}^{\infty} \left(\frac{p!}{(k+1)!} - \frac{(p-1)!}{k!} \right) = - \sum_{k=p}^{\infty} \frac{(p-1)!(k+1-p)}{(k+1)!} \\ &> - \sum_{k=p}^{\infty} \frac{1}{(k+1)k} = -\frac{1}{p}. \end{aligned}$$

Hence

$$\lfloor e(p! - (p-1)!) \rfloor + 1 = p! \sum_{k=0}^p \frac{1}{k!} - (p-1)! \sum_{k=0}^{p-1} \frac{1}{k!} \equiv - \sum_{k=0}^{p-2} \frac{(p-1)!}{k!} \pmod{p},$$

completing the proof.

Also solved by SLAVKO SIMIĆ, Belgrade, Serbia and the proposer.